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STABILITY ANALYSIS OF STOCHASTICALLY PARAMETERED NONCONSERVATIVE COLUMNS

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Abstract—Nonconservatively loaded columns, which have stochastically distributed material property values and stochastic loadings in space are considered. Young's modulus and mass density are treated to constitute random fields. The support stiffness coefficient and tip follower load are considered to be random variables. The fluctuations of external and distributed loadings are considered to constitute a random field. The variational formulation is adopted to get the differential equation and boundary conditions. The non self-adjoint operators are used at the boundary of the regularity domain. The statistics of vibration frequencies and modes are obtained using the standard perturbation method, by treating the fluctuations to be stochastic perturbations. Linear dependence of vibration and stability parameters over property value fluctuations and loading fluctuations are assumed. Bounds for the statistics of vibration frequencies are obtained. The critical load is first evaluated for the averaged problem and the corresponding eigenvalue statistics are sought. Then, the frequency equation is employed to transform the eigenvalue statistics to critical load statistics. Specialization of the general procedure to Beck, Leipholz and Pfluger columns is carried out. For Pfluger column, nonlinear transformations are avoided by directly expressing the critical load statistics in terms of input variable statistics.

1. INTRODUCTION

With the advancement of space mechanics and rocketry, the study of the stability behaviour of rockets, missiles, etc., has gained considerable importance. The inertia forces are always directed in the opposite direction to the motion whereas the drag forces arising from fluid frictional effect are always tangential to the deformed axis of the column, if the missile or rocket is idealized as a column. Such nonconservative loadings give rise to the definition of the dynamic stability criterion. Since the system lacks adjacent configurations of static equilibrium, in the presence of follower forces, the method of small oscillations is to be adopted for determining its stability behaviour. A detailed theory for such nonconservative systems can be obtained from Bolotin (1963), Ziegler (1968), Herrmann (1967) and Leipholz (1980).

This class of structures is more important and sensitive to any deviations from an idealized mathematical model as in Laudiero *et al.* (1991), Pedersen and Seyranian (1983), Pedersen (1977), Bolotin and Zhinzher (1969), Smith and Herrmann (1972), Sundararajan (1974), Sugiyama *et al.* (1974), Kounadis (1980, 1981, 1983) and Plaut and Infante (1970a,b). Recently, the probabilistic description of strength parameters, material properties, geometric boundary conditions and external loadings is gaining much momentum. A witness to the recognition of this fact is the recent spurt in the research activity (Bolotin, 1967, 1989; Tsubaki and Bazant, 1982; Soong and Cozzarelli, 1967; Zhu, 1988; Schueller and Shinozuka, 1987; Shinozuka, 1987; Vanmarcke, 1983; Augusti *et al.*, 1981; Hoshiya and Shah, 1971; Augusti *et al.*, 1984; Shinozuka and Lenoe, 1976).

The usage of modern construction materials like RCC in civil engineering and fibre-reinforced composites in aerospace engineering has not only underlined the inclusion of uncertain parameters in the analysis procedures, but also demands the optimization of design variables in a stochastic environment. In this regard the works by Jozwiak (1985, 1986) can be cited. Moreover, the condition monitoring techniques are also being developed wherein the eigensolutions play a key role (Pye and Adams, 1982; Gudmundson, 1984).

To serve the purpose of monitoring, the deviations of such eigensolutions when the system parameters are uncertain, need to be studied in detail. In the works by Boyce (1968), Soong and Cozzarelli (1976), Vom Scheidt and Purkert (1983) and Ibrahim (1987) random eigenvalue problems are considered in detail. In the area of stochastic stability analysis, research activity is directed along the following two lines: (1) deterministic systems subjected to random loading in time which is a classical random vibration problem (Wedig, 1977; Kozin, 1988; Herrmann, 1971; Pi et al., 1971; Seide, 1986; Ariaratnam and Xie, 1988; Ariratnam, 1967, 1971; Plaut and Infante, 1970a,b), and (2) stochastically parametered and conservatively loaded systems (Collins and Thomson, 1969; Shinozuka and Astill, 1972). Nonconservative, non-self-adjoint stochastic systems need to be studied, as can be seen from the literature survey.

In this paper, a general analysis is presented for stochastic systems subjected to stochastic circulatory forces, to obtain the statistics of critical loads for such systems. Specialization of this analysis is affected for Beck, Leipholz and Pfluger columns.

The stochastic fluctuations of Young's modulus and mass per unit length are treated as one-dimensional, univariate, homogeneous stochastic fields, spatially distributed. The tangential loading over the column is viewed as a one-dimensional, univariate, homogeneous stochastic field in space as the individual value deviations are stochastic.

2. SYSTEM DESCRIPTION

The mass distribution is represented by

$$m(x) = \bar{m}[1+b(x)], \tag{1}$$

where x is the spatial variable, $\bar{m}$ is the mean value of the stochastic process representing the mass distribution and $b(x)$ is a spatially distributed, one-dimensional, univariate, homogeneous, zero mean stochastic field representing the deviations or fluctuations of mass distributions about its mean value. Similarly, the Young's modulus and the distributed nonconservative loadings are described as,

$$E(x) = \bar{E}[1+a(x)], \tag{2}$$

$$g(x) = \bar{g}[1+d(x)], \tag{3}$$

where $\bar{E}$ and $\bar{g}$ are the mean values of the random processes E and g respectively. $a(x)$ and $d(x)$ are two independent, one-dimensional, univariate, zero mean, stochastic fields which are also homogeneous, representing the fluctuations of E and g distribution about their respective mean values. The autocorrelation functions of $a(x)$, $b(x)$ and $d(x)$ are given by

$$R_{aa}(z) = \langle a(x)\cdot a(x+z)\rangle, \tag{4}$$

$$R_{bb}(z) = \langle b(x)\cdot b(x+z)\rangle, \tag{5}$$

$$R_{dd}(z) = \langle d(x)\cdot d(x+z)\rangle, \tag{6}$$

where $R_{aa}(z)$, $R_{bb}(z)$ and $R_{dd}(z)$ are autocorrelation functions of random processes $a(x)$, $b(x)$ and $d(x)$, respectively. The power spectral density functions $S_{aa}(f)$, $S_{bb}(f)$ and $S_{dd}(f)$ are given through Wiener–Khintchine relations. In the above, z is the separation distance and f is the frequency.

The tip tangential nonconservative axial loading is given as

$$P = \bar{P}[1+c], \tag{7}$$

where c is a random variable with zero mean and variance σ_c^2, and $\bar{P}$ is the mean value of P. The elastic support parameter is given as

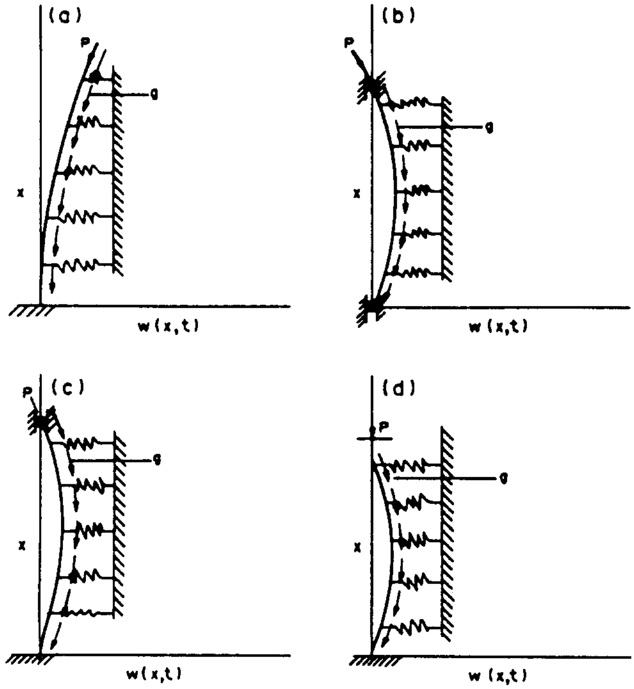

Fig. 1. Column configurations with follower loadings and Winkler foundation.

$$K = \bar{K}[1+k], \tag{8}$$

where k is a zero mean random variable with σ_k^2 as its variance. $\bar{K}$ is the mean value of K.

The extended variational principle (Leipholz, 1980) reads as

$$\int_{T_0}\left[\int_v \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tau}{\partial w_t}\right)\delta w+\delta u-Q[w]+F(x,t)\delta w\right\}\mathrm{d}v\right.$$
$$\left.-\int_{R_L}(\tau^*(w)\cdot \mathbf{D}^*(\delta w)+\tau^*(\delta w)\cdot \mathbf{D}^*(w))\,\mathrm{d}R-\int_{R_F} P[w]\delta w\,\delta R\right]\mathrm{d}t=0, \tag{9}$$

where

- x is the spatial coordinate,
- t is time,
- T_0 is a time interval,
- v is the volume of the system,
- R is the surface consisting of R_L, a supported part and R_F, a free part,
- τ is the kinetic energy density so that $T=\int_v \tau\,\mathrm{d}v$, where
- T is the total K.E. of the system,
- u is the potential energy density so that $U=\int_v u\,\mathrm{d}v$, where
- U is the total P.E. of the system,
- δw is a virtual displacement,
- $Q[w]$ represents those volume forces that do not have a potential,
- $P[w]$ represents those surface forces that are without a potential,
- $F(x,t)$ is a prescribed driving force,
- τ^* is the "vector" of those internal forces that become apparent in R_L, if that part of the surface is released from its constraints,
- $\mathbf{D}^*$ is the "vector" of corresponding displacements,
- w_t $=\partial w/\partial t$, deflection velocity, w is the deflection.

For the columns shown in Fig. 1, we have,

$$\tau = \tfrac{1}{2}\bar{m}(1+b(x))\dot{w}^2, \quad \text{where} \quad \dot{w} = \frac{\partial w}{\partial t} = w_t, \tag{10}$$

$$u = \tfrac{1}{2}\{\bar{E}(1+a(x))I(w'')^2 - \bar{g}[1+d(x)](L-x)(w')^2 - \bar{P}[1+c](w')^2 - \bar{K}[1+k]w^2\}, \tag{11}$$

where the prime denotes differentiation with respect to x, w is the lateral displacement, w_t, the deflection velocity.

We also have,

$$P[w] = -\bar{P}(1+c)(w'). \tag{12}$$

So, we get,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \tau}{\partial w_t}\right] = \bar{m}(1+b(x))\cdot\ddot{w} \tag{13}$$

and

$$\frac{\delta u}{\delta w} = \bar{P}(1+c)w'' + \bar{g}(1+d(x))[(L-x)\cdot w']' + [\bar{E}(1+a(x))\cdot I\cdot w'']'' - \bar{K}(1+k)w. \tag{14}$$

Following Leipholz (1980), introducing the following functions

$$\phi_1 = -\bar{g}(1+d(x))(L-x)w' - \bar{P}(1+c)w' - [\bar{E}(1+a(x))I\cdot w'']', \tag{15}$$

$$\phi_2 = \frac{\partial u}{\partial w''} = \bar{E}(1+a(x))\cdot I\cdot w'', \tag{16}$$

the differential equations and boundary conditions can be derived as in the deterministic case.

The differential equation is given as:

$$\bar{m}(1+b(x))\ddot{w} + [\bar{E}(1+a(x))Iw'']'' + \bar{g}(1+d(x))[(L-x)\cdot w']' + \bar{P}(1+c)w'' + \bar{g}(1+d(x))w' + Kw = F(x,t). \tag{17}$$

The boundary conditions can also be separately generated. For example, we can get the following set of boundary conditions for the column shown in Fig. 1(a):

$$w(\mathrm{o},t) = w'(\mathrm{o},t) = 0, \tag{18}$$

$$\bar{E}[1+a(L)]Iw''(L,t) = [\bar{E}(1+a(x))Iw''(x,t)]'|_{x=L} = 0. \tag{19}$$

In addition, we are having the initial conditions about w and $\dot{w}$ at the time origin.

The differential operators of eqn (17) are non-self-adjoint, i.e.unsymmetric. But there is a domain in the plot of vibration frequency vs nonconservative load parameter, wherein the differential operators behave as their self-adjoint counterparts. This domain is classically known as the "Regularity domain" (Leipholz, 1986). Physically, any point corresponding to the region beyond this "Regularity domain" represents the load parameter, which causes the amplitude of vibration to increase exponentially. The states of the system with bounded amplitudes correspond to the load parameter space given by the region within the regularity domain. So, the operators can be considered at the boundary of this domain for stability investigation. In order to get the regularity domain, the variable separable solution is employed, and the eigenvalue equation $F(\omega_n^2, q) = 0$ (where ω_n^2 is the vibration frequency of the column corresponding to the load parameter q) is derived for the loading parameter q. Since stochastic quantities are involved in the differential equations, the stability limits

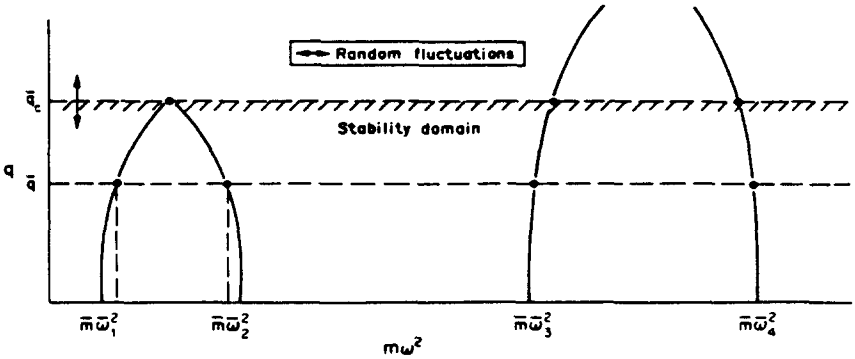

Fig. 2. General stability domain for stochastic columns subjected to stochastic follower loads.

which define the boundaries of the "Regularity domain" will have a stochastic fluctuation (see Fig. 2). The fluctuations of any boundary point on the "Regularity domain" are treated as random perturbations of the mean value of that point. This enables one to use the well-developed procedures for analysing the deterministic nonconservatively loaded columns to solve the averaged problem.

Once the "Regularity domain" is obtained, the response behaviour can be obtained through suitable methods like modal analysis (Leipholz, 1986). In the light of the above, to ascertain the stability of the system, the consideration is about the fundamental problem,

$$-\bar{m}(1+b(x))\omega_n^2 + D(w_n) = 0, \tag{20}$$

$$u_1(w_n) = 0, \tag{21}$$

where

$$D(w_n) = [\bar{E}(1+a(x))Iw_n'']'' + [\bar{g}(1+d(x))(L-x)]w_n'' + \bar{P}(1+c)w_n'' + \bar{K}(1+k)w_n \tag{22}$$

and $u_1(\cdot)$ is the boundary operator.

Letting

$$w(x,t) = \sum_{n=1} X_n(x)\cdot T_n(t),$$

on substituting into eqn (17), we will get,

$$[\bar{E}(1+a(x))IT_n(t)X_n''(x)]'' + \bar{g}(1+d(x))(L-x)X_n''(x)T_n(t) + \bar{P}(1+c)X_n''(x)T_n(t) + \bar{K}(1+k)X_n(x)T_n(t) + \bar{m}(1+b(x))X_n(x)\ddot{T}_n(t) = 0. \tag{23}$$

Since

$$\ddot{T}_n(t) = -\omega_n^2\cdot T_n(t) \tag{24}$$

using the normalized coordinate $\xi = x/L$ and cancelling $T_n(t)$ throughout, we get,

$$\frac{1}{L^4}[\bar{E}(1+a(\xi L))IX_n''(\xi)]'' + \frac{1}{L^2}\cdot[X_n''(\xi)\cdot\bar{g}(1+d(\xi L))\cdot(L-\xi L)] + \frac{1}{L^2} \times \bar{P}(1+c)X_n''(\xi) + \bar{K}(1+k)X_n(\xi) - \bar{m}(1+b(\xi L))\omega_n^2 X_n(\xi) = 0. \tag{25}$$

This is a linear differential equation with variable coefficients.

As

$$d\xi = \frac{dx}{L}, \quad \frac{d}{dx} = \frac{1}{L}\cdot\frac{d}{d\xi},$$

i.e. Jacobian is "L".

Multiplying eqn (25) by L^4 and dividing by $\bar{E}I$, we get

$$\{[1+\alpha a(\xi)]X_n''(\xi)\}'' + \bar{g}_L G(1-\xi)(1+\gamma d(\xi))X_n''(\xi) + \bar{P}G(1+c)X_n''(\xi) + \eta\bar{K}(1+k)X_n(\xi) = \lambda_n(1+\beta b(\xi))X_n(\xi), \quad (26)$$

where

$$\eta = \frac{L^4}{\bar{E}I}; \quad G = \frac{L^2}{\bar{E}I}; \quad \lambda_n = \frac{\omega_n^2\cdot\bar{m}L^4}{\bar{E}I} \quad \text{and} \quad \bar{g}_L = \bar{g}L$$

and α, β and γ are perturbation parameters.

To characterize the stochastic quantities in terms of perturbations as we discussed above, we introduce α, β and γ to be associated with $a(\xi)$, $b(\xi)$ and $d(\xi)$ into the differential equation. At the end of the analysis α, β and γ can be set equal to unity.

Employing the expansions

$$\lambda_n = \lambda_0 + \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 + c\lambda_4 + k\lambda_5 + \cdots \quad (27)$$

and

$$X_n(\xi) = X_0(\xi) + \alpha X_1(\xi) + \beta X_2(\xi) + \gamma X_3(\xi) + cX_4(\xi) + kX_5(\xi) + \cdots \quad (28)$$

the differential eqn (26) becomes

$$\begin{aligned}
&\{[1+\alpha a(\xi)][X_0''(\xi) + \alpha X_1''(\xi) + \beta X_2''(\xi) + \gamma X_3''(\xi) + cX_4''(\xi) + kX_5''(\xi)]\}'' \\
&\quad + \bar{g}_L G(1-\xi)(1+\gamma d(\xi))\{X_0''(\xi) + \alpha X_1''(\xi) + \beta X_2''(\xi) + \gamma X_3''(\xi) + cX_4''(\xi) \\
&\quad + kX_5''(\xi) + \cdots\} + \bar{P}G(1+c)\{X_0''(\xi) + \alpha X_1''(\xi) + \beta X_2''(\xi) + \gamma X_3''(\xi) \\
&\quad + cX_4''(\xi) + kX_5''(\xi) + \cdots\} + \bar{K}(1+k)\eta\{X_0(\xi) + \alpha X_1(\xi) + \beta X_2(\xi) \\
&\quad + \gamma X_3(\xi) + cX_4(\xi) + kX_5(\xi) + \cdots\} = \{\lambda_0 + \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 + c\lambda_4 + k\lambda_5 + \cdots\} \\
&\quad \times\{X_0(\xi) + \alpha X_1(\xi) + \beta X_2(\xi) + \gamma X_3(\xi) + cX_4(\xi) + kX_5(\xi) + \cdots\}\{1+\beta b(\xi)\}. \quad (29)
\end{aligned}$$

From this, the generating solution can be obtained as satisfying the following equation:

$$X_0''''(\xi) + \bar{g}_L G X_0''(\xi) - \bar{g}_L G\xi X_0''(\xi) + \bar{P}GX_0''(\xi) + \bar{K}\eta X_0(\xi) = \lambda_0 X_0(\xi). \quad (30)$$

We can also obtain the following system of equations containing the perturbations of λ_0 and $X_0(\xi)$:

$$X_1''''(\xi) + (\bar{g}_L G - \bar{g}_L\cdot G\xi)X_1''(\xi) + a(\xi)X_0''''(\xi) + \bar{P}GX_1''(\xi) + 2a'(\xi)X_0'''(\xi) + a''(\xi)X_0''(\xi) + \bar{K}\eta X_1(\xi) = \lambda_0 X_1(\xi) + \lambda_1 X_0(\xi), \quad (31)$$

$$X_2''''(\xi) + (\bar{g}_L G - \bar{g}_L G\xi)X_2''(\xi) + \bar{P}GX_2''(\xi) + \bar{K}\eta X_2(\xi) = \lambda_0 X_2(\xi) + \lambda_0 X_0(\xi)b(\xi) + \lambda_2 X_0(\xi), \quad (32)$$

$$X_3''''(\xi)+(\bar{g}_L G-\bar{g}_L G\xi)(X_3''(\xi)+d(\xi)X_o''(\xi))+\bar{P}GX_3''(\xi)+\bar{K}\eta X_3(\xi) = \lambda_o X_3(\xi)+\lambda_3 X_o(\xi). \tag{33}$$

$$X_4''''(\xi)+(\bar{g}_L G-\bar{g}_L G\xi)X_4''(\xi)+\bar{P}G(X_4''(\xi)+X_o''(\xi))+\bar{K}\eta X_3(\xi) = \lambda_o X_4(\xi)+\lambda_4 X_o(\xi). \tag{34}$$

$$X_5''''(\xi)+(\bar{g}_L G-\bar{g}_L G\xi)X_5''(\xi)+\bar{P}GX_5''(\xi)+\bar{K}\eta(X_5(\xi)+X_o(\xi)) = \lambda_o X_5(\xi)+\lambda_5 X_o(\xi). \tag{35}$$

Particular cases

(a) *Beck column.* The boundary conditions are given below:

$$\begin{aligned} w(0,t) &= 0, & X_n(o) &= 0, \\ w'(0,t) &= 0, & X_n'(o) &= 0, \\ EIw''(1,t) &= 0, & X_n''(1) &= 0, \\ [EIw''(1,t)]' &= 0, & X_n'''(1) &= 0. \end{aligned}$$

Considering eqn (30), by setting $K = g = 0$, the solution of $X_o(\xi)$ is as follows:

$$X_o(\xi) = C_1 \sin r_1\xi + C_2 \cos r_1\xi + C_3 \sinh r_1\xi + C_4 \cosh r_2\xi, \tag{36}$$

where

$$r_1 = \sqrt{\frac{p}{2}+\sqrt{\left(\frac{p}{2}\right)^2+\omega^2}}, \tag{37}$$

$$r_2 = \sqrt{-\frac{p}{2}+\sqrt{\left(\frac{p}{2}\right)^2+\omega^2}}, \tag{38}$$

$$p = \bar{P}G. \tag{39}$$

Further, C_1, C_2, C_3 and C_4 are constants.

The characteristic equation is now given by the condition,

$$\begin{vmatrix} r_1 \sin r_1 + r_2 \sinh r_2 & r_1^2 \cos r_1 + r_2^2 \cosh r_2 \\ -r_1^2 \cos r_1 - r_2^2 \cosh r_2 & r_1^3 \sin r_1 - r_2^3 \sinh r_2 \end{vmatrix} = 0,$$

i.e.

$$r_1^4 + r_2^4 + r_1 r_2 (r_1^2 - r_2^2) \sin r_1 \sinh r_2 + 2 r_1^2 r_2^2 \cos r_1 \cosh r_2 = 0. \tag{40}$$

The critical load of the Beck column with averaged properties $\bar{E}$, I, $\bar{P}$ and $\bar{m}$ is given by

$$\bar{P}_* \approx \frac{2.001\pi^2 \bar{E}I}{L^2}. \tag{41}$$

Using this value of $\bar{P}_*$ and other parameters, r_1 and r_2 are evaluated. The constants C_1, C_2, C_3 and C_4 are evaluated from these values of r_1 and r_2. The expression for $X_o(\xi)$ now corresponds to the boundary point of the "Regularity domain" and it represents the deflected shape of the column at the instability point.

(b) *Leipholz column.* The boundary conditions are the same as those for the Beck column. Considering eqn (30), by setting $K = P = 0$, $X_o(\xi)$ can be solved using these boundary conditions. The expansion functions for the deterministic case (Leipholz, 1980) are employed for the averaged problem. So,

$$X_0(\xi) = \sum_{i=0}^{3} a_i M_i(\xi). \tag{42}$$

$$M_0 = 1 + \frac{\lambda}{4!}(1-\xi)^4 - \frac{3!}{7!}\cdot\frac{\lambda(\bar{g}/\bar{E}I)}{2!}\cdot(1-\xi)^7 + \frac{\lambda^2}{8!}(1-\xi)^8 + \frac{6!3!\lambda(\bar{g}/\bar{E}I)^2}{10!5!2!}(1+\xi)^{10} + \cdots, \tag{43}$$

$$M_1 = (1-\xi) + \frac{\lambda}{5!}(1-\xi)^5 - \frac{4!}{8!}\frac{\lambda(\bar{g}/\bar{E}I)}{3!}(1-\xi)^8 + \frac{\lambda^2}{9!}(1-\xi)^9 + \cdots, \tag{44}$$

$$M_2 = (1-\xi)^2 - \frac{2!}{5!}\lambda(\bar{g}/\bar{E}I)\cdot(1-\xi)^5 + \frac{2!}{6!}\lambda(1-\xi)^6 + \frac{2!4!}{8!3!}\lambda\cdot(\bar{g}/\bar{E}I)^2 \cdot(1-\xi)^8 - \frac{2!}{9!}\lambda\cdot(\bar{g}/\bar{E}I](1-\xi)^9 + \cdots, \tag{45}$$

$$M_3 = (1-\xi)^3 - \frac{2!3!}{6!}\lambda(\bar{g}/\bar{E}I)(1-\xi)^6 + \frac{3!}{7!}\cdot\lambda\cdot(1-\xi)^7 + \frac{2!3!5!}{9!4!} \cdot\lambda(\bar{g}/\bar{E}I)^2(1-\xi)^9 - \frac{(2!3!5!+3!6!)}{10!5!}\cdot\lambda\cdot(\bar{g}/\bar{E}I)(1-\xi)^{10} + \cdots. \tag{46}$$

a_0, a_1 and a_2 are constants of the mode shape.

From these, the flutter or divergent loads can be obtained for each case, accordingly.

(c) *Pfluger column*. The differential equation is the same as that of the Leipholz column but the boundary conditions need to be modified as follows:

$$w(0,t) = 0, \quad X_0(0) = 0,$$
$$w''(0,t) = 0, \quad X_0''(0) = 0,$$
$$w(1,t) = 0, \quad X_0(1) = 0,$$
$$w''(1,t) = 0, \quad X_0''(1) = 0.$$

The general behaviour of this column considering the averaged parameters is shown in Fig. 3. The critical load is the averaged divergent load, i.e. Euler buckling case, even though $\bar{g}$

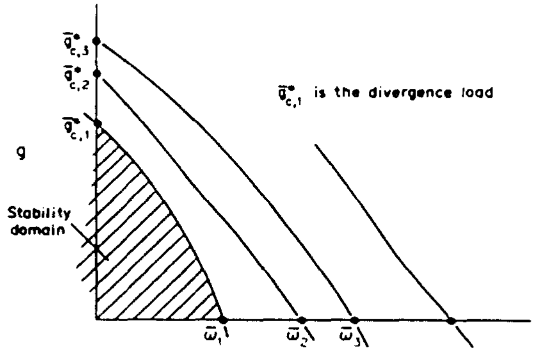

Fig. 3. Typical frequency equation plot for averaged Pfluger column.

is nonconservative. For this column the static method can be employed. So, the statistics of vibration frequencies are not needed to get the critical load statistics.

3. EIGENVALUE STATISTICS

Considering the equation for $X_1(\xi)$, multiplying by $X_o(\xi)$ and then integrating between 0 and 1, we get,

$$\int_0^1 a''(\xi)\cdot X_o''(\xi)\cdot X_o(\xi)\,d\xi+2\int_0^1 a'(\xi)\cdot X_o'''(\xi)\cdot X_o(\xi)\,d\xi+\int_0^1 X_1''''(\xi)$$
$$\cdot X_o(\xi)\,d\xi+\int_0^1 a(\xi)\cdot X_o''''(\xi)\cdot X_o(\xi)\,d\xi+\int_0^1 \bar{g}_L\cdot G\cdot X_1''(\xi)\cdot X_o(\xi)\,d\xi$$
$$-\int_0^1 \bar{g}_L G\xi X_1''(\xi)X_o(\xi)\,d\xi+\int_0^1 \bar{P}GX_1''(\xi)\cdot X_o(\xi)\,d\xi+\int_0^1 \bar{K}\eta\cdot X_1(\xi)$$
$$\cdot X_o(\xi)\,d\xi-\int_0^1 \lambda_1 X_o^2(\xi)\,d\xi-\int_0^1 \lambda_o X_1(\xi)\cdot X_o(\xi)\,d\xi=0. \tag{47}$$

Substituting eqn (30) into eqn (47), we get

$$\lambda_1=\frac{\int_0^1 a''(\xi)\cdot X_o(\xi)\cdot X_o(\xi)\,d\xi+2\int_0^1 a'(\xi)\cdot X_o'''(\xi)\cdot X_o(\xi)\,d\xi+\int_0^1 a(\xi)\cdot X_o''''(\xi)X_o(\xi)\,d\xi}{\int_0^1 X_o^2(\xi)\,d\xi}. \tag{48}$$

Adopting a similar procedure, we get

$$\lambda_2=\frac{-\lambda_o\int_0^1 b(\xi)\cdot X_o^2(\xi)\,d\xi}{\int_0^1 X_o^2(\xi)\,d\xi}, \tag{49}$$

$$\lambda_3=\frac{\bar{g}_L G\int_0^1 d(\xi)X_o''(\xi)X_o(\xi)\,d\xi}{\int_0^1 X_o^2(\xi)\,d\xi}+\cdots, \tag{50}$$

$$\lambda_4=\frac{\bar{P}G\int_0^1 X_o''(\xi)X_o(\xi)\,d\xi}{\int_0^1 X_o^2(\xi)\,d\xi} \tag{51}$$

and

$$\lambda_5=\bar{K}\eta. \tag{52}$$

These values are substituted in the λ_n expansion, so that

$$\lambda_n = \lambda_o + \frac{1}{\int_o^1 X_o^2(\xi)\,\mathrm{d}\xi}\left\{\int_o^1 a''(\xi)\cdot X_o''(\xi)\cdot X_o(\xi)\cdot \mathrm{d}\xi + 2\int_o^1 a'(\xi)\cdot X_o'''(\xi)\right.$$

$$\left.\cdot X_o(\xi)\,\mathrm{d}\xi + \int_o^1 a(\xi)\cdot X_o''''(\xi)\cdot X_o(\xi)\cdot \mathrm{d}\xi\right\} - \frac{\lambda_o}{\int_o^1 X_o^2(\xi)\,\mathrm{d}\xi}$$

$$\cdot\int_o^1 b(\xi)\cdot X_o^2(\xi)\,\mathrm{d}\xi + c\cdot\frac{\bar{P}G\left\{\int_o^1 X_o''(\xi)\cdot X_o(\xi)\,\mathrm{d}\xi\right\}}{\int_o^1 X_o^2(\xi)\,\mathrm{d}\xi}$$

$$+\frac{\int_o^1 (\bar{g}_1 G - \bar{g}_1 G\xi)\,\mathrm{d}(\xi) X_o''(\xi) X_o(\xi)\,\mathrm{d}\xi}{\int_o^1 X_o^2(\xi)\,\mathrm{d}\xi} + \bar{K}\eta\cdot k + \cdots. \tag{53}$$

As $a(\xi)$, $b(\xi)$ and $d(\xi)$ are zero mean stochastic processes, and further k and c are zero mean random variables, their derivative processes are also zero mean processes. Hence the expected value of λ_n is

$$\langle \lambda_n \rangle = \lambda_o. \tag{54}$$

The covariance between any two normalized frequencies is given by

$$C_{ij} = \bar{\lambda}[\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}+\mathrm{VII}+\mathrm{VIII}+\mathrm{IX}+\mathrm{X}+\mathrm{XI}+\text{higher order terms}]. \tag{55}$$

The detailed expressions for the different terms in the above can be found in the Appendix.

Thus, the complete covariance matrix between λs can be constructed. From here onwards the frequency corresponding to the critical load is written as λ_n for clarity.

Since

$$\mathrm{Var}(\lambda_n) = E(\lambda_n^2) - [E(\lambda_n)]^2 = E(\lambda_n^2) - \lambda_o^2, \tag{56}$$

we get from eqn (53),

$$\mathrm{Var}(\lambda_n) = \frac{1}{\left\{\int_o^1 X_o^2(\xi)\,\mathrm{d}\xi\right\}^2}\int_o^1\int_o^1\int_{-\infty}^{+\infty} S_{aa}(f)\cdot e^{if(\xi_1-\xi_2)}[X_o''(\xi_1)]^2$$

$$\cdot[X_o''(\xi_2)]^2\,\mathrm{d}f\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 + \frac{\lambda_o^2}{\left\{\int_o^1 X_o^2(\xi)\cdot\mathrm{d}\xi\right\}^2}\cdot\int_o^1\int_o^1\int_{-\infty}^{+\infty} S_{bb}(f)$$

$$\cdot e^{if(\xi_1-\xi_2)} X_o(\xi_1)\cdot X_o(\xi_1) X_o(\xi_2) X_o(\xi_2)\,\mathrm{d}f\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 + \frac{(\bar{g}_L G)^2}{\left\{\int_o^1 X_o^2(\xi)\,\mathrm{d}\xi\right\}^2}$$

$$\times\int_o^1\int_o^1\int_{-\infty}^{+\infty} S_{dd}(f)\cdot e^{if(\xi_1-\xi_2)} X_o''(\xi_1) X_o(\xi_1) X_o''(\xi_2) X_o(\xi_2)$$

$$\times (1-\xi_1)(1-\xi_2)\,\mathrm{d}f\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 + \frac{1}{\left\{\int_0^1 X_o^2(\xi)\,\mathrm{d}\xi\right\}^2}\cdot(\bar{P}G)^2\sigma_c^2$$

$$\times\left\{\int_0^1 X_o''(\xi)X_o(\xi)\,\mathrm{d}\xi\right\}^2 + K\eta\sigma_k^2 + \cdots. \tag{57}$$

This is the variance of the vibration frequency of the column at the point of instability. Numerical integration is preferable as closed form integration is tedious.

It is obvious that the physical properties E, m and loading $g(x)$ are independent and so the cross correlations are zero. Corresponding terms are automatically removed from the above expressions. So, we only have

$$R_{a'a''}(z_1 - z_2) = \langle a'(z_1)\cdot a''(z_2)\rangle, \tag{58}$$

$$R_{aa'}(z_1 - z_2) = \langle a(z_1)\cdot a'(z_2)\rangle, \tag{59}$$

$$R_{aa''}(z_1 - z_2) = \langle a(z_1)a''(z_2)\rangle, \tag{60}$$

involved in covariance relationships.

4. BOUNDS FOR EIGENVALUE STATISTICS

Having established the covariances and variances in terms of spectral density functions (or autocorrelation functions), we can proceed to establish the bounds for covariances and variances. Here, the bounds for variances are established, for clarity. In the sequel, $\rho_{aa}(\xi_1-\xi_2)$, $\rho_{bb}(\xi_1-\xi_2)$ and $\rho_{dd}(\xi_1-\xi_2)$ denote the correlation functions and $s_{aa}(f)$, $s_{bb}(f)$ and $s_{dd}(f)$ denote the normalized power spectral density functions of the respective stochastic fields.

For a given set of values of σ_c^2 and σ_k^2, it is very clear that the variance of any eigenvalue is a function of power spectral densities S_{aa}, S_{bb} and S_{dd}, for a system with known variances of material properties and loadings. If each of the stochastic fields describing the material property fluctuations and loadings has a perfect correlation regardless of the physical separation, i.e. $\rho_{aa}(\xi_1-\xi_2) = \rho_{bb}(\xi_1-\xi_2) = \rho_{dd}(\xi_1-\xi_2) = 1$, the normalized spectral densities $s_{aa}(f)$, $s_{bb}(f)$ and $s_{dd}(f)$ are direct delta functions and furthermore, $s_{aa}(f)$, $s_{bb}(f)$ and $s_{dd}(f)$ concentrate around the point $f = 0$. In this case, the variance of any eigenvalue becomes,

$$\mathrm{Var}(\lambda_n) = \frac{1}{\left\{\int_0^1 X_o^2(\xi)\,\mathrm{d}\xi\right\}^2}\left\{\sigma_a^2\left[\int_0^1 [X_o''(\xi)]^2\,\mathrm{d}\xi\right]^2 + \lambda_o^2\sigma_b^2\left[\int_0^1 X_o^2(\xi)\,\mathrm{d}\xi\right]^2\right.$$

$$\left. + (\bar{g}_L G)^2\sigma_b^2\left[\int_0^1 (1-\xi)X_o''(\xi)X_o(\xi)\,\mathrm{d}\xi\right]^2 + (\bar{P}G)^2\sigma_c^2\left[\int_0^1 X_o''(\xi)X_o(\xi)\,\mathrm{d}\xi\right]^2\right\} + K\eta\sigma_k^2 + \cdots. \tag{61}$$

The other extreme is to consider a perfectly random case, which is known to be a white noise. In that case, the correlation function is a spike function at the zero separation distance and the normalized spectral density function is a straight line parallel to the wave frequency axis. If all three fields are simultaneously considered to be the white noise fields, we have,

$$\left.\begin{aligned} s_{aa}(f) &= s_{ao} = \frac{1}{2f_u} \\ s_{hh}(f) &= s_{ho} = \frac{1}{2f_u} \\ s_{dd}(f) &= s_{do} = \frac{1}{2f_u} \end{aligned}\right] \quad \text{as} \quad f_u \rightarrow \infty . \tag{62}$$

Three different cutoff frequencies can also be used, i.e. f_{u1}, f_{u2} and f_{u3} for s_{aa}, s_{hh} and s_{dd} respectively (the limiting case that $f_u \rightarrow 0$ was discussed above). The corresponding correlation functions are, in the limit,

$$\rho_{aa}(\xi) = \delta(o), \quad \rho_{hh}(\xi) = \delta(o), \quad \rho_{dd}(\xi) = \delta(o). \tag{63}$$

Using these, the variances of eigenvalues can be obtained easily. But, these correlation functions result in infinite total power in the wave frequency domain of the random fields. So, a realistic model will account for the spectral density function as a finite power white noise or band-limited white noise. In such cases, we have

$$\rho_{aa}(\xi_1-\xi_2) = 2s_{ao}\frac{\sin f_u(\xi_1-\xi_2)}{(\xi_1-\xi_2)},$$

$$\rho_{hh}(\xi_1-\xi_2) = 2s_{ho}\frac{s\, f_u(\xi_1-\xi_2)}{(\xi_1-\xi_2)}$$

and

$$\rho_{dd}(\xi_1-\xi_2) = 2s_{do}\frac{\sin f_u(\xi_1-\xi_2)}{(\xi_1-\xi_2)}. \tag{64}$$

Now, using eqn (57), we will get

$$\begin{aligned} \text{Var}(\lambda_n) = \frac{1}{\left\{\int_0^1 X_o^2(\xi)\,d\xi\right\}^2}&\left\{2\sigma_a^2 s_{ao}\int_0^1\int_0^1 \frac{\sin f_u(\xi_1-\xi_2)}{(\xi_1-\xi_2)}[X_o''(\xi_1)]^2\right. \\ &\times [X_o''(\xi_2)]^2\,d\xi_1\,d\xi_2 + 2\lambda_o^2\sigma_h^2 s_{ho}\int_0^1\int_0^1 \frac{\sin f_u(\xi_1-\xi_2)}{(\xi_1-\xi_2)}X_o^2(\xi_1) \\ &\times X_o^2(\xi_2)\,d\xi_1\,d\xi_2 + 2(\bar{g}_L G)^2\sigma_h^2 s_{do}\int_0^1\int_0^1 \frac{\sin f_u(\xi_1-\xi_2)}{(\xi_1-\xi_2)}(1-\xi_1)(1-\xi_2) \\ &\times X_o''(\xi_1)X_o(\xi_1)X_o''(\xi_2)X_o(\xi_2\,d\xi_1\,d\xi_2 + (\bar{P}G)^2\sigma_c^2 \\ &\times \left.\left[\int_0^1 X_o''(\xi)X_o(\xi)\,d\xi\right]^2\right\} + \bar{K}\eta\sigma_k^2 + \cdots . \end{aligned} \tag{65}$$

However, exponential correlation with one parameter can also be assumed wherein the first-order autoregressive models could be accommodated to evaluate the bounds. Now, it is very clear that considering the limiting case of this sine correlation and also considering σ_c^2 and σ_k^2 to be equal to zero, the lower bound can be shown to be equal to zero.

5. STATISTICS OF CRITICAL LOADS

For the stability analysis of deterministic nonconservatively loaded columns, it has been proved that one may proceed with the so-called fundamental problem (Leipholz, 1980). The same is assumed to hold good for the averaged columns. Since the fluctuations of any point on the boundary of the "Regularity domain" are viewed as random perturbations consituting an ensemble, stability conclusions are drawn about the averaged problem response.

The averaged eigenvalue equation is used to determine the critical load q_c of the averaged column. The corresponding eigenvalue is analysed for the statistical description as above. Since, the load parameter is a function of vibration frequency related through the eigenvalue which is now rewritten for the stochastic system as

$$q_c = F_1(\lambda), \tag{66}$$

for $q = q^*$, we have

$$F(\lambda, q^*) = 0. \tag{67}$$

This equation will yield the eigenvalue $\lambda_n(q^*)$. Using the value, vibration modes are obtained and thus the averaged eigensolutions are now available.

If we know the individual and joint statistics of eigenvalues, it is possible to derive the statistical description of load parameters through explicit analytical relationships that are of closed form. So, if the monotonic unique relationship between the eigenvalues and critical load parameter exists, we can have the inverse relationship,

$$\lambda = F_2(q_c), \tag{68}$$

where

$$R_2(\cdot) = F_1^{-1}(\cdot). \tag{69}$$

In such cases, closed form analytical relationships can be written for the probability density function of q_c, using the standard transformation procedures of random variables as

$$f_{q_c}(q) = f_\lambda(\omega) \cdot \frac{\mathrm{d}}{\mathrm{d}q_c} F_2(q_c). \tag{70}$$

Similarly, the distribution function of q_c can be shown to be

$$Fq_c(q_1, q_2, q_3, \ldots, q_n) = F[\{F_1(\lambda_1) \leqslant q_1\}, \ldots, \{F_1(\lambda_n) \leqslant q_n\}]$$
$$= F_\lambda(F_2(q_1), F_2(q_2), \ldots, F_2(q_n)). \tag{71}$$

For the Leipholz column, we have the following frequency equation:

$$M_o(0) \cdot M_1'(o) - M_1(o) \cdot M_o'(o) = 0, \tag{72}$$

and for the Beck column, we have

$$p^2 + 2\omega^2 + p\omega \sin r_1 \sinh r_2 + 2\omega^2 \cos r_1 \cosh r_2 = 0 \tag{73}$$

which are obtained using the clamped-free boundary conditions. These can be solved numerically for a specific set of structural parameters. Simulation procedures can be employed to obtain critical load statistics, based on these equations.

We know, from the dynamic stability theory, that Beck and Leipholz columns are flutter systems (i.e. vibration increases in amplitude). The Pfluger column however is a pseudo-conservative system which is a conservative system of the second kind. For such columns, the kinetic method of stability investigation need not be used for the averaged problem. Euler's method can be used. Consider the Pfluger column. The critical loads can be directly obtained as follows. By setting the fundamental vibration frequency to zero, we are representing the static Euler buckling case. A parameter μ_n is introduced into eqn (17) so that,

$$\bar{g}_L = \frac{\mu_o L^2}{\bar{E}I}. \tag{74}$$

$$\mu_n = \mu_o + \alpha\mu_1 + \beta\mu_2 + \gamma\mu_3 + \cdots \quad (\text{for } k = 0), \tag{75}$$

and

$$X_n(\xi) = X_o(\xi) + \alpha X_1(\xi) + \beta X_2(\xi) + \gamma X_3(\xi). \tag{76}$$

Adopting a similar procedure that is used for obtaining the statistics of vibration frequencies, we get the following expression for the buckling parameter μ_n. For instance, if only E is random we get

$$\mathrm{Var}(\mu_n) = \frac{1}{\left\{\int_0^1 [X_o'(\xi)]^2 \, d\xi\right\}} \int_0^1 \int_0^1 \int_{-\infty}^{+\infty} S_{aa}(f)\, e^{if(\xi_1-\xi_2)} \{X_o''(\xi_1)\}^2 \{X_o''(\xi_2)\}^2 \, df \, d\xi_1 \, d\xi_2. \tag{77}$$

In this manner, the statistical information about the critical loads can very easily be obtained. It may be noted that the nonlinear transformation involved in the Beck and Leipholz columns does not appear in the Pfluger case, even though it is subjected to nonconservative loadings.

6. CONCLUSIONS

The variational formulation is adopted to get the differential equation and boundary conditions of a stochastically parametered and stochastically excited column. The perturbation method is used where the perturbing terms are taken to be the resultants of stochastic quantities of the system. The non self-adjoint operators are used within the regularity domain where the behaviour is entirely self-adjoint. First, the critical load is calculated using the averaged problem and the corresponding eigenvalue statistics are sought. Then, using the frequency equation, the transformation is performed to get the load parameter statistics, through the explicit analytical relationship, which can be solved numerically. For the Pfluger column, a direct method of evaluating the statistics of critical load which does not involve a nonlinear transformation is illustrated. Evaluation of bounds for free-vibration frequency variability enables the evaluation of bounds for critical load variability, which is an important practical information. This is so because the evaluation of exact correlation structures of the input random fields are seldom possible.

The foregoing has also enabled the development of the complete covariance structure of the frequencies and critical loads of stochastic columns which is lacking even for the simple conservative cases treated in the literature so far (Hoshiya and Shah, 1971; Shinozuka and Astill, 1972).

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APPENDIX

$$\mathrm{I} = \int_0^1 \int_0^1 \int_{-\infty}^{+\infty} f^4 \cdot S_{uu}(f)\, e^{if(\xi_1-\xi_2)}\, X_0''(\xi_1) X_0''(\xi_2) X_0(\xi_1) X_0(\xi_2)\, df\, d\xi_1\, d\xi_2.$$

$$\mathrm{II} = 4\int_0^1 \int_0^1 \int_{-\infty}^{+\infty} f^2 \cdot S_{uu}(f)\, e^{if(\xi_1-\xi_2)}\, X_0'''(\xi_1) X_0''(\xi_2) X_0(\xi_1) X_0(\xi_2)\, df\, d\xi_1\, d\xi_2.$$

$$\mathrm{III} = \int_0^1 \int_0^1 \int_{-\infty}^{+\infty} S_{uu}(f)\, e^{if(\xi_1-\xi_2)}\, X_0''''(\xi_1) X_0''''(\xi_2) X_0(\xi_1) X_0(\xi_2)\, df\, d\xi_1\, d\xi_2.$$

$$\mathrm{IV} = 2\int_0^1 \int_0^1 R_{aa'}(\xi_1-\xi_2) X_0'''(\xi_1) X_0''(\xi_2) X_0(\xi_1) X_0(\xi_2)\, d\xi_1\, d\xi_2.$$

$$\mathrm{V} = \int_0^1 \int_0^1 R_{aa'}(\xi_1-\xi_2) X_0''''(\xi_1) X_0''(\xi_2) X_0(\xi_1) X_0(\xi_2)\, d\xi_1\, d\xi_2.$$

$$\mathrm{VI} = 2\int_0^1 \int_0^1 R_{aa'}(\xi_1-\xi_2) X_0''(\xi_1) X_0'''(\xi_2) X_0(\xi_1) X_0(\xi_2)\, d\xi_1\, d\xi_2.$$

$$\mathrm{VII} = 2\int_0^1 \int_0^1 R_{aa'}(\xi_1-\xi_2) X_0''''(\xi_1) X_0'''(\xi_2) X_0(\xi_1) X_0(\xi_2)\, d\xi_1\, d\xi_2.$$

$$\mathrm{VIII} = \int_0^1 \int_0^1 R_{aa'}(\xi_1-\xi_2) X_0''(\xi_1) X_0''''(\xi_2) X_0(\xi_1) X_0(\xi_2)\, d\xi_1\, d\xi_2.$$

$$\mathrm{IX} = 2\int_0^1 \int_0^1 R_{aa}(\xi_1-\xi_2) X_0'''(\xi_1) X_0''''(\xi_2) X_0(\xi_1) X_0(\xi_2)\, d\xi_1\, d\xi_2.$$

X, XI = Similar terms for the effects of $b(x)$, $d(x)$ and c.

$$\bar{\lambda} = 1 \Big/ \left\{ \int_0^1 \int_0^1 X_0^2(\xi_1) X_0^2(\xi_2)\, d\xi_1\, d\xi_2 \right\}.$$